

TEST ELEMENTS, RETRACTS AND AUTOMORPHIC ORBITS

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Abstract. Let A_2 be a free associative or polynomial algebra of rank two over a field K of characteristic zero. Based on the degree estimate of Makar-Limanov and J.-T. Yu, we prove: 1) An element $p \in A_2$ is a test element if p does not belong to any proper retract of A_2 ; 2) Every endomorphism preserving the automorphic orbit of a nonconstant element of A_2 is an automorphism.

1. INTRODUCTION AND MAIN RESULTS

In the sequel, K always denotes a field of characteristic zero. Automorphisms (endomorphisms) always mean K -automorphisms (K -endomorphisms).

Let A_n be a free associative or polynomial algebra of rank n over K . An element $p \in A_n$ is called a *test element* if every endomorphism of A_n fixing p is an automorphism. A subalgebra R of A_n is called a *retract* if there is an idempotent endomorphism $\pi(\pi^2 = \pi)$ of A_n (called a *retraction* or a *projection*) such that $\pi(A_n) = R$. Test elements and retracts of groups and other algebras are defined in a similar way. Test elements and retracts of algebras and groups have recently been studied in [3, 5, 6, 7, 12, 18, 20, 21, 22, 23, 24, 29, 30, 32, 33].

A test element does not belong to any proper retract for any algebra or group as the corresponding non-injective idempotent endomorphism is not an automorphism. The converse is proved by Turner [34] for free groups, by Mikhalev and Zolotykh [24] and by Mikhalev and J. -T. Yu [21, 22] for free Lie algebras and free Lie superalgebras respectively,

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and by Mikhalev, Umirbaev and J. -T. Yu [19] for free nonassociative algebras. See also Mikhalev, Shpilrain and J. -T. Yu [18].

In view of the above, we may raise the following

Conjecture 1. If an element $p \in A_n$ does not belong to any proper retract of A_n , then p is a test element.

Recently, V.Shpilrain and J. -T. Yu [33] proved Conjecture 1 for $\mathbb{C}[x, y]$. A key lemma in their proof is the degree estimate of Shestakov and Umirbaev [26], which plays a crucial role in the recent celebrated solution of the Nagata conjecture [27, 28] and the Strong Nagata conjecture [35].

More recently, Makar-Limanov and J. -T. Yu [16] developed a new combinatorial method based on the Lemma on radicals and obtained a sharp degree estimate for the ‘free’ case, namely, for a free associative algebra or a polynomial algebra over a field of characteristic zero. It has found applications for automorphisms and coordinates of polynomial and free associative algebras. See S.-J. Gong and J.-T. Yu [9].

Now we consider another related problem. In an algebra or a group, certainly an automorphism preserves the automorphic orbit of an element p . The converse is proved by Shpilrain [31] and Ivanov [10] for free groups of rank two, by D. Lee [14] for free groups of any rank, by Mikhalev and J. -T. Yu [22] for free Lie algebras and by Mikhalev, Umirbaev and J.-T. Yu [19] for free non-associative algebras, by van den Essen and Shpilrain [7] for A_2 when p is a coordinate, by Jelonek [11] for polynomial algebras over \mathbb{C} when p is a coordinate. For the related linear coordinate preserving problem, see, for instance, S.-J. Gong and J.-T. Yu [8]. See also the book [18].

In view of the above, we may raise the following

Conjecture 2. Let $p \in A_n - K$. Then any endomorphism of A_n preserving the automorphic orbit of p must be an automorphism.

Conjecture 2 has recently been settled affirmatively by J.-T. Yu [36] for $A_2 = \mathbb{C}[x, y]$ based on Shpilrain and J.-T.Yu’s characterization of test elements of $\mathbb{C}[x, y]$ in [33] and the main result in Drensky and J.-T.Yu [6].

In this paper, based on the recent degree estimate of Makar-Limanov and J.-T.Yu [16], the main ideals and techniques in Drensky and J.-T.Yu [6], Shpilrain and J.-T. Yu [32, 33], and J.-T.Yu [36], we prove both Conjecture 1 and Conjecture 2 for $n = 2$. Our main results are

Theorem 1.1. *If an element $p \in A_2$ does not belong to any proper retract of A_2 , then p is a test element of A_2 .*

Theorem 1.1 was proved by Shpilrain and J.-T.Yu [33] for $A_2 = \mathbb{C}[x, y]$.

Theorem 1.2. *If an endomorphism ϕ of A_2 preserves the automorphic orbit of a nonconstant element $p \in A_2$, then ϕ is an automorphism of A_2 .*

Theorem 1.2 was proved by J.-T.Yu [36] for $A_2 = \mathbb{C}[x, y]$.

Crucial to the proofs of the above two theorems are the following two results, which have their own interests.

Theorem 1.3. *Let $p \in A_2$ have outer rank two. Then any injective endomorphism ϕ of A_2 is an automorphism if $\phi(p) = p$.*

Theorem 1.3 may be viewed as an analogue of a result in Turner [34] for free groups. It was proved for $A_2 = \mathbb{C}[x, y]$ in J.-T.Yu [36] based on a result in Shpilrain and J.-T.Yu [33].

Theorem 1.4. *An element $p(x, y) \in A_2$ belongs to a proper retract of A_2 if $p(x, y)$ is fixed by a noninjective endomorphism ϕ of A_2 . Moreover, in this case there exists a positive integer m such that ϕ^m is a retraction of A_2 .*

Theorem 1.4 was proved for $A_2 = \mathbb{C}[x, y]$ in Drensky and J.-T.Yu [6].

2. PROOFS

The following two lemmas are Theorem 1.1 and Proposition 1.2 in Makar-Limanov and J.-T.Yu [16].

Lemma 2.1. *Let $A_n = K\langle x_1, \dots, x_n \rangle$ be a free associative algebra over a field K of characteristic zero, $f, g \in A$ be algebraically independent, f^+ and g^+ are algebraically independent, or f^+ and g^+ are algebraically dependent and neither $\deg(f) \mid \deg(g)$ nor $\deg(g) \mid \deg(f)$, $p \in K\langle x, y \rangle$. Then*

$$\deg(p(f, g)) \geq \frac{\deg[f, g]}{\deg(fg)} w_{\deg(f), \deg(g)}(p).$$

Here \deg is the total degree, $w_{\deg(f), \deg(g)}(p)$ is the weighted degree of p when the weight of the first variable is $\deg(f)$ and the weight of the second variable is $\deg(g)$, f^+ and g^+ are the highest homogeneous components of f and g respectively, and $[f, g] = fg - gf$ is the commutator of f and g .

Lemma 2.2. *Let $A_n = K[x_1, \dots, x_n]$ be a polynomial algebra over a field K of characteristic zero, $f, g \in A$ be algebraically independent, $p \in K[x, y]$. Then*

$$\deg(p(f, g)) \geq w_{\deg(f), \deg(g)}(p) \left[1 - \frac{(\deg(f), \deg(g))(\deg(fg) - \deg(J(f, g)) - 2)}{\deg(f) \deg(g)} \right].$$

Here \deg is the total degree, $w_{\deg(f), \deg(g)}(p)$ is the weighted degree of p when the weight of the first variable is $\deg(f)$ and the weight of the second variable is $\deg(g)$, $(\deg(f), \deg(g))$ is the greatest common divisor of $\deg(f)$ and $\deg(g)$, $\deg(J(f, g))$ is the largest degree of non-zero Jacobian determinants of f and g with respect to two of x_1, \dots, x_n .

The following characterization of a proper retract of A_2 was obtained by Shpilrain and J.-T. Yu [32] based on a result of Costa [3].

Lemma 2.3. *Let R be a proper retract of A_2 . Then $R = K[r]$ for some $r \in A_2$. Moreover, there exists an automorphism α of A_2 such that $\alpha(r) = x + w(x, y)$, where $w(x, y)$ belongs to the ideal of A_2 generated by y .*

Lemma 2.4. *Let $p \in A_2$ with outer rank 2 and $f, g \in A_n$. Then $w_{\deg(f), \deg(g)}(p) \geq \deg(f) + \deg(g)$. If every monomial of p contains both x and y and $\deg(p) > 2$, then $w_{\deg(f), \deg(g)}(p) > \deg(f) + \deg(g)$.*

Proof. 1) If p contains a monomial containing both x and y , where $i \neq 0, j \neq 0$, $w_{\deg(f), \deg(g)}(p) \geq i(\deg(f)) + j(\deg(g)) \geq \deg(f) + \deg(g)$. If every monomial of p contains both x and y and $\deg(p) > 2$, then the second inequality becomes strict.

2) Otherwise p must contain monomials x^i and y^j where $i \geq 2, j \geq 2$. Then $w_{\deg(f), \deg(g)}(p) \geq 2 \max\{\deg(f), \deg(g)\} \geq \deg(f) + \deg(g)$. \square

Lemma 2.5. *Let $A_n = K\langle x_1, \dots, x_n \rangle$ be a free associative algebra over an arbitrary field K of zero characteristic, $f, g \in A_2$ be algebraically independent, $p \in K\langle x, y \rangle$ have outer rank two. Then*

$$\deg(p(f, g)) \geq \deg[f, g].$$

If every monomial of p contains both x and y and $\deg(p) > 2$, then

$$\deg(p(f, g)) > \deg[f, g].$$

Proof. Let 1) If f^+ and g^+ are algebraically independent; or f^+ , g^+ are algebraically dependent, but $\deg(f) \nmid \deg(g)$ and $\deg(g) \nmid \deg(f)$. Then by Lemma 2.1 and Lemma 2.4, $\deg(p(f, g)) \geq \deg[f, g]$. If, in addition, every monomial of p contains both x and y and $\deg(p) > 2$, then by Lemma 2.1 and Lemma 2.4, $\deg(p(f, g)) > \deg[f, g]$.

2) Otherwise there exists an automorphism α , which is the composition of a sequence of elementary automorphisms, such that $\alpha(f) = \bar{f}$, $\alpha(g) = \bar{g}$, $\bar{p} = \alpha^{-1}(p)$ satisfying the condition in 1). Then $\deg(p(f, g)) = \deg(\bar{p}(\bar{f}, \bar{g})) \geq \deg[\bar{f}, \bar{g}] = \deg[f, g]$. \square

Lemma 2.6. Let $A_n = K[x_1, \dots, x_n]$ be a polynomial algebra over an arbitrary field K of zero characteristic, $f, g \in A_n$ be algebraically independent, $p \in K[x, y]$ has outer rank two. Then

$$\deg(p(f, g)) \geq \deg(J(f, g)) + 2.$$

Proof. We may assume $\deg(f) = m$, $\deg(g) = n$. As p has outer rank 2, by Lemma 2.4 then p contains a monomial with both x and y , or contains monomials x^i and y^j where $i \geq 2$, $j \geq 2$.

1) Let f^+ and g^+ be algebraically independent.

a) If there exists a monomial in p containing both x and y , then $\deg(p(f, g)) \geq \deg(f) + \deg(g) \geq \deg(J(f, g)) + 2$;

b) Otherwise p must have a monomial of x^i where $i \geq 2$, and another monomial y^j where $j \geq 2$, then $\deg(p(f, g)) \geq 2 \max\{m, n\} \geq \deg(f) + \deg(g) \geq \deg(J(f, g)) + 2$;

2) Let f^+ , g^+ be algebraically dependent, and $m \nmid n$ and $n \nmid m$.

c) If $w_{\deg(f), \deg(g)}(p) < \text{lcm}(m, n)$, then in $p(f, g)$, f^+ and g^+ cannot cancel out, hence similar to the case 1 a), $\deg(p(f, g)) \geq \deg(f) + \deg(g) \geq \deg(J(f, g)) + 2$.

d) Otherwise $w_{\deg(f), \deg(g)}(p) \geq \text{lcm}(m, n) = mn/(m, n)$. We also have $mn = (m, n)\text{lcm}(m, n) \geq (m, n)(m + n)$. Hence $\deg(p(f, g)) \geq \deg(J(f, g)) + 2$ by Lemma 2.2.

3) Let f^+ , g^+ be algebraically dependent, but $m \mid n$ or $n \mid m$. Then by same process in the Proof 2) of Lemma 2.4, we may reduce to the above case 1) or case 2). \square

Lemma 2.7. Let $\phi = (f, g)$ be an injective endomorphism of $K\langle x, y \rangle$ but not an automorphism. Then $\deg([\phi^k(x), \phi^k(y)]) \geq k + 2$ for $k \geq 0$.

Proof. We use induction. $\deg[\phi^0(x), \phi^0(y)] = \deg[x, y] = 0 + 2$. Assuming $\deg[\phi^{k-1}(x), \phi^{k-1}(y)] \geq (k-1) + 2$. Define $p(x, y) := [f(x, y), g(x, y)]$. As $\phi = (f, g)$ is injective, every monomial of $p(x, y)$ contains both x and y . Since $\phi = (f, g)$ is not an automorphism, by the well-known result of Dicks (see, Dicks [4], or Cohn [2]), $\deg(p(x, y)) > \deg(x) + \deg(y) = 2$. Applying Lemma 2.5, $\deg(p(u, v)) > \deg[u, v]$ for $u = \phi^{k-1}(x)$, $v = \phi^{k-1}(y)$, hence $\deg[\phi^k(x), \phi^k(y)] = \deg(p(\phi^{k-1}(x), \phi^{k-1}(y))) > \deg[\phi^{k-1}(x), \phi^{k-1}(y)] \geq (k-1) + 2 = k + 1$. Therefore, $\deg[\phi^k(x), \phi^k(y)] \geq (k+1) + 1 = k + 2$. \square

Lemma 2.8. *Let $\phi = (f, g)$ be an injective endomorphism of $K[x, y]$ but not an automorphism and there exists an element $p \in K[x, y]$ fixed by ϕ . Then $\deg(J(\phi^k(x), \phi^k(y))) \geq k$ for $k \geq 0$.*

Proof. As ϕ fixes p , ϕ is not an automorphism, by a result of Kraft [13] (see also Shpilrain and J.-T. Yu [32]), $\deg(J(\phi(x), \phi(y))) = \deg(J(f, g)) \geq 1$. By the chain rule for the Jacobian,

$$\begin{aligned} & \deg(J(\phi^k(x), \phi^k(y))) \\ &= \deg(J(f, g)(\phi^{k-1}(x), \phi^{k-1}(y))(J(\phi^{k-1}(x), \phi^{k-1}(y)))) \\ &\geq \deg(J(\phi^{k-1}(x), \phi^{k-1}(y))) + 1. \end{aligned}$$

The proof is concluded by induction. \square

Lemma 2.9. *Let $\phi = (f, g)$ be an injective endomorphism of A_2 but not an automorphism. Then any element $p \in A_2$ with outer rank 2 cannot be fixed by ϕ .*

Proof. If $p \in A_2$ with outer rank two fixed by ϕ , then $\deg(p(f, g)) = \deg(p(\phi^k(x), \phi^k(y))) \geq k + 2$ for all $k \geq 0$. by Lemma 2.5 and Lemma 2.7 for noncommutative case; and by Lemma 2.6 and Lemma 2.8 for polynomial case. The contradiction completes the proof. \square

Proof of Theorem 1.3.

By Lemma 2.9. \square

Proof of Theorem 1.4.

The proof presented here is similar to the proof of the main Theorem in Drensky and J.-T. Yu [6].

Let $p \in A_2 - \{0\}$ fixed by a noninjective endomorphism of A_2 . Then $\phi(x)$ and $\phi(y)$ are algebraically dependent over K . Let us denote the image of $\phi(A_2)$ by $S = K[\phi(x), \phi(y)]$ (since $\phi(x)$ and $\phi(y)$ are algebraically dependent, $\phi(x)$ and $\phi(y)$ are in a polynomial algebra of

rank one over K as a consequence of a result of Bergman [1] for non-commutative case and as a consequence of a result of Shestakov and Umirbaev [26] for polynomial case) and by $Q(S)$ the field of fractions of S . Therefore the transcendence degree of $Q(S)$ over K is 1. Let $0 \neq q(x, y) \in (\text{Ker}(\phi)) \cap S$. Since $p(x, y)$ also belongs to S , the polynomials p and q are algebraically dependent and

$$h(p, q) = a_0(q)p^n + a_1(q)p^{n-1} + \dots + a_{n-1}(q)p + a_n(q) = 0$$

for an irreducible polynomial $h(u, v) \in K[u, v]$ and $a_i(t) \in K[t]$, $i = 0, 1, \dots, n$. Hence $\phi(h(p, q)) = h(\phi(p), \phi(q)) = h(p, 0)$,

$$a_0(0)p^n + a_1(0)p^{n-1} + \dots + a_{n-1}(0)p + a_n(0) = 0.$$

Therefore $a_0(0) = a_1(0) = \dots = a_n(0) = 0$. Now the polynomials $a_i(t)$ have no constant terms and $h(u, v)$ is divisible by v which contradicts to the irreducibility of $h(u, v)$. Therefore $(\text{Ker}(\phi)) \cap S = 0$ and ϕ acts injectively on its image S . Hence we may extend the action of ϕ on $Q(S)$ (because $a_1/b_1 = a_2/b_2$ in $Q(S)$ is equivalent to $a_1b_2 = a_2b_1$ and hence $\phi(a_1/b_1) = \phi(a_1)/\phi(b_1) = \phi(a_2)/\phi(b_2) = \phi(a_2/b_2)$). By Lüroth's theorem (See, for instance, Schinzel [25]), $Q(S) = K(w)$ for some $w \in Q(S)$. The automorphism ϕ fixes $p(x, y)$ and its extension $\bar{\phi}$ on $Q(S)$ fixes $K(p)$. Since w is algebraic over $K(p)$, $Q(S)$ is a finite dimensional vector space over $K(p)$ and $\bar{\phi}$ is a $K(p)$ -linear operator of $Q(S)$ with trivial kernel. Hence $\bar{\phi}$ is invertible on $Q(S)$ and we may consider $\bar{\phi}$ as an automorphism of the finite field extension $Q(S)$ over $K(p)$ which fixes $K(p)$. By Galois theory ($\bar{\phi}$ interchanges the roots of the minimal polynomial of w over $K(p)$ and there are finite number of possibilities for $\bar{\phi}(w)$), $\bar{\phi}$ has finite order. Let $\bar{\phi}^m = 1$. Then $\phi^{m+1}(r) = \phi^m(\phi(r)) = \bar{\phi}^m(\phi(r)) = \phi(r)$ for every $r \in A_2$ and $(\phi^m)^2 = \phi^{m+1}\phi^{m-1} = \phi\phi^{m-1} = \phi^m$. Therefore $\pi = \phi^m$ is a retraction (idempotent endomorphism) of A_2 with a nontrivial kernel and $\pi(p) = p$. Hence $p(x, y)$ is in the image of π which is a proper retract $\pi(A_2)$ of A_2 . \square

Proof of Theorem 1.1.

As $p \in A_2$ does not belong to any proper retract of A_2 , by Theorem 1.4, any endomorphism ϕ of A_2 fixing p must be injective. By Lemma 2.3, obviously p must have outer rank two, otherwise p would belong to a proper retract of A_2 . By Theorem 1.3, ϕ is an automorphism. Hence p is a test element of A_2 . \square

Proof of Theorem 1.2.

The proof presented here is similar to the proof of the main result Theorem 1.4 in J.-T. Yu [36].

We may assume that $\phi(p) = p$. By the definition of the test element, we may assume p is not a test element. By Theorem 1.1, we may assume p belongs to a proper retract $K[r]$ of A_2 . By a result in J.-T. Yu [36], we may assume p has outer rank 2. By Theorem 1.3, we may assume ϕ is non-injective. Suppose that $p = f(r)$, where $f \in K[t] - K$, $\deg(f) = m$. By Theorem 1.4, $\pi = \phi^m$ is a retraction of A_2 to $K[r]$. As ϕ preserves the automorphic orbit of p , so does $\pi = \phi^m$. Applying Lemma 2.3 (suppose $\alpha(r) = x + w(x, y)$, where $w(x, y) \notin K[y]$ belongs to the ideal of A_2 generated by y , α is some automorphism of A_2 , replace r by $\alpha(r)$, and π by $\alpha\pi\alpha^{-1}$), we have reduced our proof to the following

Lemma 2.10. *Let $r = x + w(x, y)$, where $w(x, y)$ belongs to the ideal of A_2 generated by y and $w(x, y) \notin K[y]$, π the retraction of A_2 onto $K[r]$ defined by $\pi(x) = x + w(x, y)$, $\pi(y) = 0$, $f \in K[t] - K$. Then π does not preserve the automorphic orbit of $f(r)$.*

Proof. Suppose on the contrary, π preserves the automorphic orbit of $f(r)$. Then for any automorphism α of A_2 , $\pi\alpha(f(r)) = \beta(f(r)) \in K[r]$ for some automorphism β of A_2 . Note that $\pi\beta(f(r)) = \pi^2\beta(f(r)) = \pi\alpha(f(r)) = \beta(f(r))$. By Theorem 1.4, $\pi^{\deg(f)} = \pi$ is the retraction of A_2 onto the retract $K[\beta(r)]$ taking $\beta(r)$ to $\beta(r)$. By hypothesis, π is also a retraction of A_2 onto the retract $K[r]$ taking r to r . This forces that $\beta(r) = cr + d$ for some $c \in K^*$, $d \in K$. We have concluded that for any automorphism α of A_2 , there exists some $c \in K^*$, $d \in K$, such that $\pi\alpha(f(r)) = f(cr + d)$.

Now we proceed the proof in two cases.

1. Noncommutative case: $A_2 = K\langle x, y \rangle$.

Denote by \mathcal{C} the commutator ideal of $K\langle x, y \rangle$.

a) If $w(x, y) \in \mathcal{C}$, then take α to be the automorphism of $K\langle x, y \rangle$ defined by $\alpha(x) = y + x^2$, $\alpha(y) = x$. Direct calculation shows that $\pi\alpha(f(r)) = f(r^2 + w(r^2, r)) = f(r^2) \neq f(cr + d)$, a contradiction.

b) If $w(x, y) \notin \mathcal{C}$, then $w^a(x, y) = yv(x, y)$ for some $v(x, y) \in K[x, y] - \{0\}$. Here $w^a(x, y) \in K[x, y]$ is the image of $w(x, y)$ under the abelianization from $K\langle x, y \rangle$ onto $K[x, y]$. Let M be a positive integer greater than $\deg(v(x, y))$, it is easy to see that $x^M - y$ does not

divide $v(x, y)$ in $K[x, y]$. Let α be the automorphism of $K\langle x, y \rangle$ defined by $\alpha(x) = x$, $\alpha(y) = y + x^M$. Then $\pi\alpha(f(r)) = f(r + w(r, r^M)) = f(r + r^M v(r, r^M))$. As $x^M - y$ does not divide $v(x, y)$, $v(r, r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M v(r, r^M)) \neq f(cr + d)$, a contradiction.

2. Polynomial case: $A_2 = K[x, y]$.

In this case we write $w(x, y) = yq(x, y)$ where $q(x, y) \notin K[y]$. Let M be a positive integer greater than $\deg(q(x, y))$, it is easy to see that $x^M - y$ does not divide $q(x, y)$ in $K[x, y]$. Let α be the automorphism of $K[x, y]$ defined by $\alpha(x) = x$, $\alpha(y) = y + x^M$. Then easy calculation shows that $\pi\alpha(f(r)) = f(r + r^M q(r, r^M))$. As $x^M - y$ does not divide $q(x, y)$, $q(r, r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M q(r, r^M)) \neq f(cr + d)$. The contradiction completes the proof. \square

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